SOME TECHNIQUES ON NONLINEAR ANALYSIS AND APPLICATIONS

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Abstract. In this paper we present two different results in the context of nonlinear analysis. The first one is essentially a nonlinear technique that, in view of its strong generality, may be useful in different practical problems. The second result, more technical, but also connected to the first one, is an extension of the well-known Pietsch Domination Theorem. The last decade witnessed the birth of different families of Pietsch Domination-type results and some attempts of unification. Our result, that we call "full general Pietsch Domination Theorem" is potentially a definitive Pietsch Domination Theorem which unifies the previous versions and delimits what can be proved in this line. The connections to the recent notion of weighted summability are traced.

1. Introduction and motivation

Common, even simple, mathematical problems usually involve nonlinear maps, sometimes acting on sets with little (or none) algebraic structure; so the extension of linear techniques to the nonlinear setting, besides its intrinsic mathematical interest, is an important task for potential applications. In fact it is mostly a challenging task, since linear arguments are commonly ineffective in a more general setting. The following problem illustrates this situation.

If X, Y are Banach spaces, $u, v : X \to Y$ are continuous linear operators, $C > 0$ and $1 \leq p \leq q < \infty$ it is possible to show that:

1.-) If

$$\sum_{j=1}^{m} \|u(x_j)\|^p \leq C \sum_{j=1}^{m} \|v(x_j)\|^p \text{ for every } m \text{ and all } x_1, ..., x_m \in X, \tag{1.1}$$

then

$$\sum_{j=1}^{m} \|u(x_j)\|^q \leq C \sum_{j=1}^{m} \|v(x_j)\|^q \text{ for every } m \text{ and all } x_1, ..., x_m \in X. \tag{1.2}$$

Also, in the same direction:

2.-) If

$$\sum_{j=1}^{m} \|u(x_j)\|^p \leq C \sup_{\varphi \in B_{X^*}} \sum_{j=1}^{m} |\varphi(x_j)|^p \text{ for every } m \text{ and all } x_1, ..., x_m \in X, \tag{1.3}$$

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then

$$\sum_{j=1}^{m}\|u(x_j)\|^q \leq C \sup_{\varphi\in B_{X^*}} \sum_{j=1}^{m} |\varphi(x_j)|^q \text{ for every } m \text{ and all } x_1,...,x_m \in X, \tag{1.4}$$

where X^* is the topological dual of X and B_{X^*} denotes its closed unit ball. More generally, if

$$\begin{array}{c} p_j \leq q_j \text{ for } j=1,2,\\ 1\leq p_1 \leq p_2 < \infty,\\ 1 \leq q_1 \leq q_2 < \infty,\\ \frac{1}{p_1}-\frac{1}{q_1} \leq \frac{1}{p_2}-\frac{1}{q_2}, \end{array} \tag{1.5}$$

then

$$\left(\sum_{j=1}^{m}\|u(x_j)\|^{q_1}\right)^{1/q_1} \leq C \sup_{\varphi\in B_{X^*}} \left(\sum_{j=1}^{m} |\varphi(x_j)|^{p_1}\right)^{1/p_1} \text{ for every } m \text{ and all } x_1,...,x_m \in X \tag{1.6}$$

implies that

$$\left(\sum_{j=1}^{m}\|u(x_j)\|^{q_2}\right)^{1/q_2} \leq C \sup_{\varphi\in B_{X^*}} \left(\sum_{j=1}^{m} |\varphi(x_j)|^{p_2}\right)^{1/p_2} \text{ for every } m \text{ and all } x_1,...,x_m \in X. \tag{1.7}$$

Problem 1.1. *What about nonlinear versions of the above results? Are there any?*

Problem 1.2. *What about nonlinear versions in which the spaces X and Y are just sets, with no structure at all?*

The interested reader can find the proof of the implication (1.6)⇒(1.7) in [23, p. 198]. This result was essentially proved by S. Kwapień in 1968 (see [35]) and it is what is now called "Inclusion Theorem for absolutely summing operators". A quick look shows that the linearity is fully explored and a nonlinear version of this result, if there is any, would require a whole new technique. It is worth mentioning that practical problems may also involve sets with less structure than Banach spaces (or less structure than linear spaces or even than metric spaces) and a "full" nonlinear version (with no structure on the spaces involved) would certainly be interesting for potential applications.

In this direction we will prove a very general result, which we will call "Inclusion Principle", which, due its extreme generality, may be useful in different contexts, even outside of pure mathematical analysis. The arguments used in the proof of the "Inclusion Principle" are, albeit tricky, fairly clear and simple in nature, but we do believe this technique may be useful in different contexts. To illustrate its reach, at least in the context of Functional Analysis, we show that very particular cases of the Inclusion Principle can contribute to the nonlinear theory of absolutely summing operators.

Below, as an illustration, we describe an extremely particular case of the forthcoming Inclusion Principle:

Let X be an arbitrary non-void set and Y be a normed space; suppose that p_j and q_j satisfy (1.5). If $f, g : X \to Y$ are arbitrary mappings and there is a constant

$C > 0$ so that

$$\sum_{j=1}^{m} \|f(x_j)\|^{q_1} \leq C \sum_{j=1}^{m} \|g(x_j)\|^{p_1},$$

for every m and all $x_1, \ldots, x_m \in X$, then there is a constant $C_1 > 0$ such that

$$\left(\sum_{j=1}^{m} \|f(x_j)\|^{q_2} \right)^{\frac{1}{\alpha}} \leq C_1 \sum_{j=1}^{m} \|g(x_j)\|^{p_2}$$

for every m and all $x_1, \ldots, x_m \in X$, with

$$\alpha = \frac{q_2 p_1}{q_1 p_2} \text{ if } p_1 < p_2.$$

The case $p_1 = p_2$ is trivial. The parameter α is a kind of adjustment, i.e., the price that one has to pay for the complete lack of linearity, and precisely when $p_j = q_j$ for $j = 1$ and 2 we have $\alpha = 1$ and no adjustment is needed. In other words, the parameter α indicates the necessary adjustments (in view of the lack of linearity) when p_j and q_j become distant.

The second main contribution of this paper is more technical, but also useful. It is what we call “full general Pietsch Domination Theorem” which, as will be shown, has several applications and seems to be a definitive answer to the attempt of delimiting the amplitude of action of Pietsch Domination-type theorems.

The Pietsch Domination Theorem (PDT) (sometimes stated as the Pietsch Factorization Theorem) was proved in 1967 by A. Pietsch, in his classical paper [57], and since then it has played a special and important role in Banach Space Theory having its honour place in several textbooks related to Banach Space Theory [3, 19, 23, 58, 61, 64]; PDT has a strong connection with the aforementioned inclusion results, as we explain below. In fact, if $0 < p < \infty$, PDT states that for a given continuous linear operator $u : X \to Y$ the following assertions are equivalent:

(i) There exists a $C > 0$ so that

$$\sum_{j=1}^{m} \|u(x_j)\|^p \leq C \sup_{\varphi \in B_{X^*}} \sum_{j=1}^{m} |\varphi(x_j)|^p \text{ for every } m.$$

(ii) There are a Borel probability measure μ on B_{X^*} (with the weak-star topology) and $C > 0$ such that

$$\|u(x)\| \leq C \left(\int_{B_{X^*}} |\varphi(x_j)|^p \, d\mu \right)^{\frac{1}{p}}. \tag{1.8}$$

Using the canonical inclusions between L_p spaces we conclude that, if $0 < p \leq q < \infty$, the inequality (1.8) implies that

$$\|u(x)\| \leq C \left(\int_{B_{X^*}} |\varphi(x_j)|^q \, d\mu \right)^{\frac{1}{q}}$$

and we obtain the implication (1.3)⇒(1.4) as a corollary.

Due to its strong importance in Banach Space Theory, PDT was re-discovered in different contexts in the last decades (e.g. [1, 17, 24, 28, 29, 40, 41, 50]) and, since 2009, in [14, 15, 53] some attempts were made in the direction of showing that one unique PDT can be stated in such a general way that all the possible Pietsch

Domination-type theorems would be straightforward particular cases of this unified Pietsch-Domination theorem.

Thus, the second contribution of this paper is to prove a "full general Pietsch Domination Theorem" that, besides its own interest, we do believe that will be useful to delimit the scope of Pietsch-type theorems. Some connections with the recent promising notion of weighted summability introduced in [52] are traced.

2. The Inclusion Principle

In this section we deal with general values for p_j and q_j satisfying (1.5). In order to be useful in different contexts, we state the result in a very general form.

Let X, Y, Z, V and W be (arbitrary) non-void sets. The set of all mappings from X to Y will be represented by $Map(X,Y)$. Let $\mathcal{H} \subset Map(X,Y)$ and

$$R\colon Z \times W \longrightarrow [0,\infty), \text{ and}$$
$$S\colon \mathcal{H} \times Z \times V \longrightarrow [0,\infty)$$

be arbitrary mappings. If $1 \leq p \leq q < \infty$, suppose that

$$\sup_{w\in W} \sum_{j=1}^{m} R\left(z_j, w\right)^p < \infty \text{ and } \sup_{v\in V} \sum_{j=1}^{m} S(f, z_j, v)^q < \infty$$

for every positive integer m and $z_1, ..., z_m \in Z$ (in most of the applications V and W are compact spaces and R and S have some trace of continuity to assure that both sup are finite). If $\alpha \in \mathbb{R}$, we will say that $f \in \mathcal{H}$ is RS-abstract $((q,\alpha),p)$-summing (notation $f \in RS_{((q,\alpha),p)}$) if there is a constant $C > 0$ so that

$$\left(\sup_{v\in V} \sum_{j=1}^{m} S(f, z_j, v)^q\right)^{\frac{1}{\alpha}} \leq C \sup_{w\in W} \sum_{j=1}^{m} R\left(z_j, w\right)^p, \tag{2.1}$$

for all $z_1, \ldots, z_m \in Z$ and m.

Theorem 2.1 (Inclusion Principle)**.** *If p_j and q_j satisfy (1.5), then*

$$RS_{((q_1,1),p_1)} \subset RS_{((q_2,\alpha),p_2)}$$

for

$$\alpha = \frac{q_2 p_1}{q_1 p_2} \text{ if } p_1 < p_2.$$

Proof. Let $f \in RS_{((q_1,1),p_1)}$. There is a $C > 0$ such that

$$\sup_{v\in V} \sum_{j=1}^{m} S(f, z_j, v)^{q_1} \leq C \sup_{w\in W} \sum_{j=1}^{m} R\left(z_j, w\right)^{p_1}, \tag{2.2}$$

for all $z_1, \ldots, z_m \in Z$ and $m \in \mathbb{N}$. If each $\eta_1, ..., \eta_m$ is a positive integer, by considering each z_j repeated η_j times in (2.2) one can easily note that

$$\sup_{v\in V} \sum_{j=1}^{m} \eta_j S(f, z_j, v)^{q_1} \leq C \sup_{w\in W} \sum_{j=1}^{m} \eta_j R\left(z_j, w\right)^{p_1}, \tag{2.3}$$

for all $z_1, \ldots, z_m \in Z$ and $m \in \mathbb{N}$. Now, using a clever argument credited to Mendel and Schechtman (used recently, in different contexts, in [28, 52, 53]) we can conclude that (2.3) holds for arbitrary positive real numbers η_j. The idea is to pass from integers to rationals by "cleaning" denominators and from rationals to real numbers using density.

Since $p_1 < p_2$ we have $q_1 < q_2$. Define p, q as

$$\frac{1}{p} = \frac{1}{p_1} - \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} - \frac{1}{q_2}.$$

So we have $1 \leq q \leq p < \infty$; next, let $m \in \mathbb{N}$ and $z_1, z_2, ..., z_m \in Z$ be fixed. For each $j = 1, ..., m$, consider the map

$$\lambda_j : V \to [0, \infty)$$
$$\lambda_j(v) := S(f, z_j, v)^{\frac{q_2}{q}}.$$

Thus,

$$\begin{aligned}\lambda_j(v)^{q_1} S(f, z_j, v)^{q_1} &= S(f, z_j, v)^{\frac{q_1 q_2}{q}} S(f, z_j, v)^{q_1} \\ &= S(f, z_j, v)^{q_2}.\end{aligned}$$

Recalling that (2.3) is valid for arbitrary positive real numbers η_j, we get, for $\eta_j = \lambda_j(v)^{q_1}$,

$$\begin{aligned}\sum_{j=1}^{m} S(f, z_j, v)^{q_2} &= \sum_{j=1}^{m} \lambda_j(v)^{q_1} S(f, z_j, v)^{q_1} \\ &\leq C \sup_{w \in W} \sum_{j=1}^{m} \lambda_j(v)^{q_1} R(z_j, w)^{p_1}\end{aligned}$$

for every $v \in V$. Also, since $p, p_2 > p_1$ and $\frac{1}{(p/p_1)} + \frac{1}{(p_2/p_1)} = 1$, invoking Hölder's Inequality we obtain

$$\begin{aligned}\sum_{j=1}^{m} S(f, z_j, v)^{q_2} &\leq C \sup_{w \in W} \sum_{j=1}^{m} \lambda_j(v)^{q_1} R(z_j, w)^{p_1} \\ &\leq C \sup_{w \in W} \left[\left(\sum_{j=1}^{m} \lambda_j(v)^{\frac{q_1 p}{p_1}} \right)^{\frac{p_1}{p}} \left(\sum_{j=1}^{m} R(z_j, w)^{p_2} \right)^{\frac{p_1}{p_2}} \right] \\ &= C \left(\sum_{j=1}^{m} \lambda_j(v)^{\frac{q_1 p}{p_1}} \right)^{\frac{p_1}{p}} \sup_{w \in W} \left(\sum_{j=1}^{m} R(z_j, w)^{p_2} \right)^{\frac{p_1}{p_2}}\end{aligned}$$

for every $v \in V$. Since $\frac{q_1 p}{p_1} \geq p \geq q$ we have $\|.\|_{\ell_{\frac{q_1 p}{p_1}}} \leq \|.\|_{\ell_q}$ and then

$$\begin{aligned}\sum_{j=1}^{m} S(f, z_j, v)^{q_2} &\leq C \left(\sum_{j=1}^{m} \lambda_j(v)^{q} \right)^{\frac{q_1}{q}} \sup_{w \in W} \left(\sum_{j=1}^{m} R(z_j, w)^{p_2} \right)^{\frac{p_1}{p_2}} \\ &= C \left(\sum_{j=1}^{m} S(f, z_j, v)^{q_2} \right)^{\frac{q_1}{q}} \sup_{w \in W} \left(\sum_{j=1}^{m} R(z_j, w)^{p_2} \right)^{\frac{p_1}{p_2}}\end{aligned}$$

for every $v \in V$. We thus have

$$\left(\sum_{j=1}^{m} S(f,z_j,v)^{q_2}\right)^{1-\frac{q_1}{q}} \leq C \sup_{w\in W}\left(\sum_{j=1}^{m} R\left(z_j,w\right)^{p_2}\right)^{\frac{p_1}{p_2}}$$

for every $v \in V$, and we can finally conclude that

$$\left(\sup_{v\in V}\sum_{j=1}^{m} S(f,z_j,v)^{q_2}\right)^{\frac{q_1p_2}{q_2p_1}} \leq C^{\frac{p_2}{p_1}} \sup_{w\in W}\sum_{j=1}^{m} R\left(z_j,w\right)^{p_2}.$$

□

Remark 2.2. *It is interesting to mention that as q_j becomes closer to p_j for $j=1$ and 2, the value $\frac{q_1p_2}{q_2p_1}$ becomes closer to 1(which occurs in the linear setting when $p_j = q_j$ for $j=1$ and 2). In other words, the effect of the lack of linearity in our estimates is weaker when p_j and q_j are closer and, in the extreme case where $p_1 = q_1$ and $p_2 = q_2$, then $\alpha = 1$ and we have a "perfect generalization" of the linear result.*

3. Applications on the nonlinear absolutely summing operators

3.1. **Absolutely summing operators: a brief summary.** In the real line it is well-known that a series is absolutely convergent precisely when it is unconditionally convergent. For infinite-dimensional Banach spaces it is easy to verify that the situation is different; for example, for ℓ_p spaces with $1 < p < \infty$, it is easy to construct an unconditionally convergent series which fails to be absolutely convergent. However the behavior for arbitrary Banach spaces was not known before 1950. For ℓ_1, for example, the construction is much more complicated (see M. S. McPhail's work from 1947, [39]).

This perspective leads to the feeling that this property (having an unconditionally summable series which is not absolutely summable) could be shared by all infinite-dimensional Banach-spaces. This question was raised by S. Banach in his monograph [5, page 40] and appears as Problem 122 in the Scottish Book (see [45]).

In 1950, A. Dvoretzky and C. A. Rogers [27] solved this question by showing that in every infinite-dimensional Banach space there is an unconditionally convergent series which fails to be absolutely convergent. This new panorama of the subject called the attention of A. Grothendieck who provided, in his thesis [31], a different approach to the Dvoretzky-Rogers result. His thesis, together with his Résumé [30], can be regarded as the beginning of the theory of absolutely (q,p)-summing operators.

The notion of absolutely (q,p)-summing operator, as we know nowadays, is due to B. Mitiagin and A. Pełczyński [48] and A. Pietsch [57]. Pietsch's paper is a classical and particular role is played by the Domination Theorem, which presents an unexpected measure-theoretical characterization of p-summing operators. The same task was brilliantly done, one year later, by J. Lindenstrauss and A. Pełczyński's paper [37] which reformulated Grothendieck's tensorial arguments giving birth to a comprehensible theory with broad applications in Banach Space Theory.

From now on the space of all continuous linear operators from a Banach space X to a Banach space Y will be denoted by $\mathcal{L}(X,Y)$. If $1 \leq p \leq q < \infty$, we say that the

Banach space operator $u : X \to Y$ is (q,p)-summing if there is an induced operator

$$\begin{array}{cccc} \hat{u} & : & \ell_p^{\text{weak}}(X) & \longrightarrow & \ell_q^{\text{strong}}(Y) \\ & & (x_n)_{n=1}^{\infty} & \mapsto & (ux_n)_{n=1}^{\infty}. \end{array}$$

Above $\ell_p^{\text{weak}}(X) := \{(x_j)_{j=1}^{\infty} \subset X : \sup_{\varphi \in B_{X^*}} (\sum_j |\varphi(x_j)|^p)^{1/p} < \infty\}$. The class of absolutely (q,p)-summing linear operators from X to Y will be represented by $\Pi_{q,p}(X,Y)$. For details on the linear theory of absolutely summing operators we refer to the classical book [23]. The linear theory of absolutely summing operators was intensively investigated in the 70's and several classical papers can tell the story (we mention [7, 8, 18, 22, 26, 47] and the monograph [23] for a complete panorama).

Special role is played by Grothendieck's Theorem and Pietsch-Domination Theorem:

Theorem 3.1 (Grothendieck). *Every continuous linear operator from ℓ_1 to ℓ_2 is absolutely $(1,1)$-summing.*

Theorem 3.2 (Lindenstrauss and Pełczyński). *If X and Y are infinite-dimensional Banach spaces, X has an unconditional Schauder basis and $\Pi_{1,1}(X,Y) = \mathcal{L}(X,Y)$ then $X = \ell_1$ and Y is a Hilbert space.*

Theorem 3.3 (Pietsch-Domination Theorem). *If X and Y are Banach spaces, a continuous linear operator $T : X \to Y$ is absolutely (p,p)-summing if and only if there is a constant $C > 0$ and a Borel probability measure μ on the closed unit ball of the dual of X, $(B_{X^*}, \sigma(X^*,X))$, such that*

$$\|T(x)\| \leq C \left(\int_{B_{X^*}} |\varphi(x)|^p \, d\mu \right)^{\frac{1}{p}}.$$

An immediate consequence of the Pietsch Domination Theorem is that, for $1 \leq r \leq s < \infty$, every absolutely (r,r)-summing operator is absolutely (s,s)-summing. However a more general result is valid. As mentioned in the first section, this result is essentially due to Kwapień ([36]):

Theorem 3.4 (Inclusion Theorem). *If X and Y are Banach spaces and*

$$\begin{array}{c} p_j \leq q_j \text{ for } j = 1,2, \\ 1 \leq p_1 \leq p_2 < \infty, \\ 1 \leq q_1 \leq q_2 < \infty, \\ \frac{1}{p_1} - \frac{1}{q_1} \leq \frac{1}{p_2} - \frac{1}{q_2}, \end{array} \tag{3.1}$$

then

$$\Pi_{q_1,p_1}(X,Y) \subset \Pi_{q_2,p_2}(X,Y). \tag{3.2}$$

The end of the 60's was also the time of the birth of the notion of *type* and *cotype*. It probably began to be conceived in the Séminaire Laurent Schwartz, and after important contributions by J. Hoffmann-Jørgensen [33], B. Maurey [46], S. Kwapień [36], and H. Rosenthal [60], the concept was formalized by B. Maurey and G. Pisier [47].

Since B. Maurey and G. Pisier's seminal paper [47], the connection of the notion of cotype and the concept of absolutely summing operators become clear. In 1992, M. Talagrand [63] proved very deep results complementing previous results of B. Maurey and G. Pisier showing that cotype 2 spaces have indeed a special behavior in the theory of absolutely summing operators:

Theorem 3.5 (Maurey-Pisier and Talagrand)**.** *If a Banach space X has cotype q, then id_X is absolutely $(q,1)$-summing. The converse is true, except for $q=2$.*

In the last two decades the interest of the theory was moved to the nonlinear setting although there are still some challenging questions being investigated in the linear setting (see [11, 16]). For example, recent results from [16] complements the Lindenstrauss-Pełczyński Theorem 3.2 (below $\cot X$ denotes the infimum of the cotypes assumed by X):

Theorem 3.6. *(*[16]*) Let X and Y be infinite-dimensional Banach spaces.*
(i) If $\Pi_{1,1}(X,Y)=\mathcal{L}(X,Y)$ then $\cot X=\cot Y=2$.
(ii) If $2\leq r<\cot Y$ and $\Pi_{q,r}(X,Y)=\mathcal{L}(X,Y)$, then $\mathcal{L}(\ell_1,\ell_{\cot Y})=\Pi_{q,r}(\ell_1,\ell_{\cot Y})$.

The extension of the classical linear theory of absolutely summing operators to the multilinear setting is very far from being a mere exercise of generalization with expected results obtained by induction. In fact, some multilinear approaches are simple but there are several delicate questions related to the multilinear extensions of absolutely summing operators. Some illustrative examples and applications can be seen in [2, 4, 21, 55, 56]). For non-multilinear approaches we refer to [9, 34, 43, 44]).

The advance of the nonlinear theory of absolutely summing operators leads to the search for nonlinear versions of the Pietsch Domination-Factorization Theorem (see, for example, [1, 13, 14, 28, 29, 40]). Recently, in [15] (see also an addendum in [53] and [52] for a related result), an abstract unified approach to Pietsch-type results was presented as an attempt to show that all the known Pietsch-type theorems were particular cases of a unified general version. However, these approaches were not complete, as we will show later.

3.2. **Applications to the theory of absolutely summing multilinear operators.** The multilinear theory of absolutely summing mappings seems to have its starting point in [6, 38] but only in the 1980's it gained more attention, motivated by A. Pietsch's work [59]; recently some nice results and applications have appeared, mainly related to the notion of fully or multiple summability (see [2, 10, 12, 20, 21] and references therein). This section will actually show that for multilinear mappings there exists an improved version of the Inclusion Principle (we just need to explore the multi-linearity).

For technical reasons the present abstract setting is slightly different from the one of the previous section. Let X, Y, V, G, W be (arbitrary) non-void sets, Z a vector space and $\mathcal{H}\subset Map(X,Y)$. Consider the arbitrary mappings

$$\begin{aligned} R&\colon Z\times G\times W\longrightarrow[0,\infty)\\ S&\colon \mathcal{H}\times Z\times G\times V\longrightarrow[0,\infty). \end{aligned}$$

Let $1 \leq p \leq q < \infty$ and $\alpha \in \mathbb{R}$. Suppose that

$$\sup_{w\in W} \sum_{j=1}^{m} R(z_j, g_j, w)^p < \infty \text{ and } \sup_{v\in V} \sum_{j=1}^{m} S(f, z_j, g_j, v)^q < \infty$$

for every positive integer m and $z_1, ..., z_m \in Z$ and $g_1, ..., g_m \in G$. We will say that $f \in \mathcal{H}$ is (q,p)-abstract (R,S)-summing (notation $f \in RS_{(q,p)}$) if there is a constant $C > 0$ so that

$$\left(\sup_{v\in V} \sum_{j=1}^{m} S(f, z_j, g_j, v)^q\right)^{\frac{1}{q}} \leq C \left(\sup_{w\in W} \sum_{j=1}^{m} R(z_j, g_j, w)^p\right)^{1/p}, \tag{3.3}$$

for all $z_1, \ldots, z_m \in Z$, $g_1, ..., g_m \in G$ and $m \in \mathbb{N}$. We will say that S and R are multiplicative in the variable Z if

$$R(\lambda z, g, w) = |\lambda| R(z, g, w),$$
$$S(f, \lambda z, g, v) = |\lambda| S(f, z, g, v).$$

Theorem 3.7. *Let p_j and q_j be as in (3.1) and suppose that S and R are multiplicative in the variable Z. Then*

$$RS_{(q_1,p_1)} \subset RS_{(q_2,p_2)}.$$

Proof. If $p_1 = p_2 = p$ the result is clear. So, let us consider $p_1 < p_2$ (and hence $q_1 < q_2$). If $f \in RS_{(q_1,p_1)}$, there is a $C > 0$ such that

$$\left(\sup_{v\in V} \sum_{j=1}^{m} S(f, z_j, g_j, v)^{q_1}\right)^{\frac{1}{q_1}} \leq C \sup_{w\in W} \left(\sum_{j=1}^{m} R(z_j, g_j, w)^{p_1}\right)^{\frac{1}{p_1}}, \tag{3.4}$$

for all $z_1, \ldots, z_m \in Z$, $g_1, ..., g_m \in G$ and $m \in \mathbb{N}$. Then

$$\left(\sup_{v\in V} \sum_{j=1}^{m} S(f, \lambda_j z_j, g_j, v)^{q_1}\right)^{\frac{1}{q_1}} \leq C \sup_{w\in W} \left(\sum_{j=1}^{m} R(\lambda_j z_j, g_j, w)^{p_1}\right)^{\frac{1}{p_1}}, \tag{3.5}$$

for all $z_1, \ldots, z_m \in Z$, $\lambda_1, ..., \lambda_m \in \mathbb{K}$, $g_1, ..., g_m \in G$ and $m \in \mathbb{N}$. Define p, q by

$$\frac{1}{p} = \frac{1}{p_1} - \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} - \frac{1}{q_2}.$$

So we have $1 \leq q \leq p < \infty$; let $m \in \mathbb{N}$, $z_1, z_2, ..., z_m \in Z$ and $g_1, ..., g_m \in G$ be fixed. For each $j = 1, ..., m$, consider

$$\lambda_j : V \to [0, \infty)$$
$$\lambda_j(v) := S(f, z_j, g_j, v)^{\frac{q_2}{q}}.$$

So, recalling that S is multiplicative in Z, we have

$$\left(\sum_{j=1}^{m} S(f, z_j, g_j, v)^{q_2}\right)^{\frac{1}{q_1}} = \left(\sum_{j=1}^{m} S(f, \lambda_j(v) z_j, g_j, v)^{q_1}\right)^{\frac{1}{q_1}}$$
$$\leq C \sup_{w\in W} \left(\sum_{j=1}^{m} R(\lambda_j(v) z_j, g_j, w)^{p_1}\right)^{\frac{1}{p_1}}$$

for every $v \in V$. Since R is multiplicative in Z and, as we did before, from Hölder's Inequality we obtain

$$
\begin{aligned}
\left(\sum_{j=1}^{m} S(f,z_j,g_j,v)^{q_2}\right)^{\frac{1}{q_1}} &\le C \sup_{w\in W}\left(\sum_{j=1}^{m} \lambda_j(v)^{p_1} R\left(z_j,g_j,w\right)^{p_1}\right)^{\frac{1}{p_1}} \\
&\le C \sup_{w\in W}\left[\left(\sum_{j=1}^{m}\lambda_j(v)^{p}\right)^{\frac{p_1}{p}}\left(\sum_{j=1}^{m} R\left(z_j,g_j,w\right)^{p_2}\right)^{\frac{p_1}{p_2}}\right]^{\frac{1}{p_1}} \\
&= C\left(\sum_{j=1}^{m}\lambda_j(v)^{p}\right)^{\frac{1}{p}} \sup_{w\in W}\left(\sum_{j=1}^{m} R\left(z_j,g_j,w\right)^{p_2}\right)^{\frac{1}{p_2}}
\end{aligned}
$$

for every $v \in V$. Since $p \ge q$ we have $\|.\|_{\ell_p} \le \|.\|_{\ell_q}$ and then

$$
\begin{aligned}
\left(\sum_{j=1}^{m} S(f,z_j,g_j,v)^{q_2}\right)^{\frac{1}{q_1}} &\le C\left(\sum_{j=1}^{m}\lambda_j(v)^{q}\right)^{\frac{1}{q}} \sup_{w\in W}\left(\sum_{j=1}^{m} R\left(z_j,g_j,w\right)^{p_2}\right)^{\frac{1}{p_2}} \\
&= C\left(\sum_{j=1}^{m} S(f,z_j,g_j,v)^{q_2}\right)^{\frac{1}{q}} \sup_{w\in W}\left(\sum_{j=1}^{m} R\left(z_j,g_j,w\right)^{p_2}\right)^{\frac{1}{p_2}}
\end{aligned}
$$

for every $v \in V$ and we easily conclude the proof. $\square$

Let us show how the above result applies to the multilinear theory of absolutely summing mappings. Our intention is illustrative rather than exhaustive. From now on we will use the notation $\mathcal{L}(X_1,...,X_n;Y)$ to represent the spaces of continuous n-linear mappings from $X_1 \times \cdots \times X_n$ to Y. For the theory of multilinear mappings between Banach spaces we refer to [25, 49]. Consider the following concepts of multilinear summability for $1 \le p \le q < \infty$ (inspired in [17, 24]):

1.-) A mapping $T \in \mathcal{L}(X_1,...,X_n;Y)$ is (q,p)-semi integral if there exists $C \ge 0$ such that

$$
\left(\sum_{j=1}^{m} \| T(x_j^1,...,x_j^n) \|^q\right)^{1/q} \le C\left(\sup_{\varphi_l \in B_{X_l^*}, l=1,...,n} \sum_{j=1}^{m} | \varphi_1(x_j^1)...\varphi_n(x_j^n) |^p\right)^{1/p} \tag{3.6}
$$

for every $m \in \mathbb{N}$, $x_j^l \in X_l$ with $l = 1,...,n$ and $j = 1,...,m$. In the above situation we write $T \in \mathcal{L}_{si(q,p)}(X_1,...,X_n;Y))$.

2.-) A mapping $T \in \mathcal{L}(X_1,...,X_n;Y)$ is strongly (q,p)-summing if there exists $C \ge 0$ such that

$$
\left(\sum_{j=1}^{m} \| T(x_j^1,...,x_j^n) \|^q\right)^{1/q} \le C\left(\sup_{\varphi \in B_{\mathcal{L}(X_1,...,X_n;\mathbb{K})}} \sum_{j=1}^{m} | \varphi(x_j^1,...,x_j^n) |^p\right)^{1/p}
$$

for every $m \in \mathbb{N}$, $x_j^l \in X_l$ with $l = 1,...,n$ and $j = 1,...,m$. In the above situation we write $T \in \mathcal{L}_{ss(q,p)}(X_1,...,X_n;Y)$.

For both concepts there is a natural Pietsch-Domination-type theorem (see [17, 24]) and as a corollary the following inclusion results hold:

Proposition 3.8. *If $1 \leq p \leq q < \infty$, then, for any Banach spaces $X_1, ..., X_n, Y$, the following inclusions hold:*

$$\mathcal{L}_{si(p,p)}(X_1, ..., X_n; Y) \subset \mathcal{L}_{si(q,q)}(X_1, ..., X_n; Y) \text{ and}$$
$$\mathcal{L}_{ss(p,p)}(X_1, ..., X_n; Y) \subset \mathcal{L}_{ss(q,q)}(X_1, ..., X_n; Y).$$

However, the Pietsch Domination Theorem is useless for the other choices of p_j, q_j. But, as it will be shown, in this case the multilinearity allows us to obtain better results than those from Theorem 2.1.

For the class of semi-integral mappings we may choose $Z = X_1$, $G = X_2 \times \cdots \times X_n$, $W = B_{X_1^*} \times \cdots \times B_{X_n^*}$, $V = \{0\}$, $\mathcal{H} = \mathcal{L}(X_1, ..., X_n; Y)$ and consider the mappings

$$R\colon Z \times G \times W \longrightarrow [0, \infty)$$
$$R\left(x_1, (x_2..., x_n), (\varphi_1, ..., \varphi_n)\right) = \mid \varphi_1(x_1)...\varphi_n(x_n) \mid$$

and

$$S\colon \mathcal{H} \times Z \times G \times V \longrightarrow [0, \infty)$$
$$S\left(T, x_1, (x_2..., x_n), 0\right) = \parallel T(x_1, ..., x_n) \parallel .$$

The case of the class of strongly summing multilinear mappings is analogous. So, as a consequence of Theorem 3.7, we have:

Proposition 3.9. *If p_j and q_j are as in (3.1) then, for any Banach spaces $X_1, ..., X_n, Y$, the following inclusions hold:*

$$\mathcal{L}_{si(q_1,p_1)}(X_1, ..., X_n; Y) \subset \mathcal{L}_{si(q_2,p_2)}(X_1, ..., X_n; Y) \text{ and}$$
$$\mathcal{L}_{ss(q_1,p_1)}(X_1, ..., X_n; Y) \subset \mathcal{L}_{ss(q_2,p_2)}(X_1, ..., X_n; Y).$$

3.3. **Applications to non-multilinear absolutely summing operators.** As in the previous section, we intend to illustrate how the Inclusion Principle can be invoked in other situations; we have no exhaustive purpose.

Let us consider the following definitions extending the notion of semi-integral and strongly multilinear mappings to the non-multilinear context, even with spaces having a less rich structure than a Banach space:

Definition 3.10. *Let $X_1, ..., X_n$ be normed spaces and $Y = (Y, d)$ be a metric space. An arbitrary map $f : X_1 \times \cdots \times X_n \to Y$ is $((q, \alpha), p)$-semi integral at $(a_1, ..., a_n) \in X_1 \times \cdots \times X_n$ (notation $f \in Map_{si((q,\alpha),p))}(X_1, ..., X_n; Y)$) if there exists $C \geq 0$ such that*

$$\left(\sum_{j=1}^{m} \left(d\left(f(a_1 + x_j^1, ..., a_n + x_j^n), f(a_1, ..., a_n) \right) \right)^q \right)^{1/\alpha}$$
$$\leq C \sup_{\varphi_l \in B_{X_l^*}, l=1,...,n} \sum_{j=1}^{m} \mid \varphi_1(x_j^1)...\varphi_n(x_j^n) \mid^p$$

for every $m \in \mathbb{N}$, $x_j^l \in X_l$ with $l = 1, ..., n$ and $j = 1, ..., m$.

Definition 3.11. *Let $X_1, ..., X_n$ be normed spaces and $Y = (Y, d)$ be a metric space. An arbitrary map $f : X_1 \times \cdots \times X_n \to Y$ is strongly $((q, \alpha), p)$-summing* *at* $(a_1, ..., a_n) \in X_1 \times \cdots \times X_n$ *(notation* $f \in Map_{ss((q,\alpha),p))}(X_1, ..., X_n; Y)$*) if there exists* $C \geq 0$ *such that*

$$\left(\sum_{j=1}^{m} \left(d\left(f(a_1 + x_j^1, ..., a_n + x_j^n), f(a_1, ..., a_n) \right) \right)^q \right)^{1/\alpha}$$
$$\leq C \sup_{\varphi \in \mathcal{L}(X_1,...,X_n;\mathbb{K})} \sum_{j=1}^{m} \mid \varphi(x_j^1, ..., x_j^n) \mid^p$$

for every $m \in \mathbb{N}$, $x_j^l \in X_l$ *with* $l = 1, ..., n$ *and* $j = 1, ..., m$.

By choosing adequate parameters in Theorem 2.1 we obtain:

Theorem 3.12. *If* p_j *and* q_j *satisfy (3.1), then*

$$Map_{si((q_1,1),p_1)}(X_1, ..., X_n; Y) \subset Map_{si((q_2,\alpha),p_2))}(X_1, ..., X_n; Y) \text{ and}$$
$$Map_{ss((q_1,1),p_1)}(X_1, ..., X_n; Y) \subset Map_{ss((q_2,\alpha),p_2))}(X_1, ..., X_n; Y)$$

for

$$\alpha = \frac{q_2 p_1}{q_1 p_2} \text{ if } p_1 < p_2.$$

3.4. **Applications to non-multilinear absolutely summing operators in the sense of Matos.** In [43] M. Matos considered a concept of summability which can be characterized by means of an inequality as follows:

If X and Y are Banach spaces, a map $f : X \to Y$ is *absolutely* (q,p)*-summing at* a if there are constants $C > 0$ and $\delta > 0$ such that

$$\sum_{j=1}^{\infty} \|f(a + z_j) - f(a)\|^q \leq C \sup_{\varphi \in B_{X^*}} \sum_{j=1}^{\infty} |\varphi(z_j)|^p,$$

for all $(z_j)_{j=1}^{\infty} \in \ell_p^u(X)$ and

$$\left\|(z_j)_{j=1}^{\infty}\right\|_{w,p} := \sup_{\varphi \in B_{X^*}} \left(\sum_{j=1}^{\infty} |\varphi(z_j)|^p \right)^{1/p} < \delta.$$

Above,

$$\ell_p^u(X) := \left\{ (z_j)_{j=1}^{\infty} \in \ell_p^{\text{weak}}(X); \lim_{n\to\infty} \left\|(z_j)_{j=n}^{\infty}\right\|_{w,p} = 0 \right\}.$$

It is worth mentioning that there exists a version of our inclusion principle in this context. If $\alpha \in \mathbb{R}$, we will say that $f : X \to Y$ is *Matos absolutely* $((q,\alpha),p)$*-summing at* a (denoted by $f \in M_{((q,\alpha),p)}$) if there are constants $C > 0$ and $\delta > 0$ such that

$$\left(\sum_{j=1}^{\infty} \|f(a + z_j) - f(a)\|^q \right)^{\frac{1}{\alpha}} \leq C \sup_{\varphi \in B_{X^*}} \sum_{j=1}^{\infty} |\varphi(z_j)|^p, \tag{3.7}$$

for all $(z_j)_{j=1}^{\infty} \in \ell_p^u(X)$ and $\left\|(z_j)_{j=1}^{\infty}\right\|_{w,p} < \delta$. If $\alpha = 1$ we recover Matos' original concept and simply write (q,p) instead of $((q,1),p)$.

With this at hand, we can now state the following result:

Theorem 3.13. *If p_j and q_j are as in* (3.1)*, then*

$$M_{(q_1,p_1)} \subset M_{((q_2,\alpha),p_2)}$$

for

$$\alpha = \frac{q_2 p_1}{q_1 p_2}$$

whenever $p_1 < p_2$.

4. A full general version of the Pietsch Domination Theorem

If $X_1, ..., X_n, Y$ are Banach spaces, the set of all continuous n-linear mappings $T : X_1 \times \cdots \times X_n \to Y$ is represented by $\mathcal{L}(X_1, ..., X_n; Y)$. All measures considered in this paper will be probability measures defined in the Borel sigma-algebras of compact topological spaces.

In this section, and for the sake of completeness, we will recall the more general version that we know, until now, for the Pietsch Domination Theorem. This approach is a combination of [15] and a recent improvement from [53] and will be generalized in the subsequent section.

Let X, Y and E be (arbitrary) non-void sets, $\mathcal{H}$ be a family of mappings from X to Y, G be a Banach space and K be a compact Hausdorff topological space. Let

$$R\colon K \times E \times G \longrightarrow [0,\infty) \text{ and } S\colon \mathcal{H} \times E \times G \longrightarrow [0,\infty)$$

be mappings so that the following property hold:

"The mapping

$$R_{x,b}\colon K \longrightarrow [0,\infty) \text{ defined by } R_{x,b}(\varphi) = R(\varphi, x, b)$$

is continuous for every $x \in E$ and $b \in G$."

Let R and S be as above and $0 < p < \infty$. A mapping $f \in \mathcal{H}$ is said to be R-S-abstract p-summing if there is a constant $C > 0$ so that

$$\left(\sum_{j=1}^{m} S(f, x_j, b_j)^p\right)^{\frac{1}{p}} \le C \sup_{\varphi \in K} \left(\sum_{j=1}^{m} R\left(\varphi, x_j, b_j\right)^p\right)^{\frac{1}{p}}, \tag{4.1}$$

for all $x_1, \ldots, x_m \in E$, $b_1, \ldots, b_m \in G$ and $m \in \mathbb{N}$.

The general unified PDT reads as follows:

Theorem 4.1. *(*[15, 53]*) Let R and S be as above, $0 < p < \infty$ and $f \in \mathcal{H}$. Then f is R-S-abstract p-summing if and only if there is a constant $C > 0$ and a Borel probability measure μ on K such that*

$$S(f, x, b) \le C \left(\int_K R\left(\varphi, x, b\right)^p d\mu\right)^{\frac{1}{p}} \tag{4.2}$$

for all $x \in E$ and $b \in G$.

From now on, if $X_1, ..., X_n, Y$ are arbitrary sets, $Map(X_1, ..., X_n; Y)$ will denote the set of all arbitrary mappings from $X_1 \times \cdots \times X_n$ to Y (no assumption is necessary).

Let $0 < q_1, ..., q_n < \infty$ and $1/q = \sum_{j=1}^{n} 1/q_j$. A map $f \in Map(X_1, ..., X_n; Y)$ is $(q_1, ..., q_n)$-dominated at $(a_1, ..., a_n) \in X_1 \times \cdots \times X_n$ if there is a $C > 0$ and there are Borel probabilities μ_k on $B_{X_k^*}$, $k = 1, ..., n$, such that

$$\left\|f(a_1 + x^{(1)}, ..., a_n + x^{(n)}) - f(a_1, ..., a_n)\right\| \leq C\prod_{k=1}^{n}\left(\int_{B_{X_k^*}} \left|\varphi(x^{(k)})\right|^{q_k} d\mu_k\right)^{\frac{1}{q_k}} \tag{4.3}$$

for all $x^{(j)} \in X_j$, $j = 1, ..., n$.

In our recent note [52] we observed that the general approach from [15, 53] was not able to characterize the mappings satisfying (4.3), and a new Pietsch-type theorem was proved:

Theorem 4.2. *([52])A map $f \in Map(X_1, ..., X_n; Y)$ is $(q_1, ..., q_n)$-dominated at $(a_1, ..., a_n) \in X_1 \times \cdots \times X_n$ if and only if there is a $C > 0$ such that*

$$\begin{aligned}(4.4)\qquad &\left(\sum_{j=1}^{m}\left(\left|b_j^{(1)}...b_j^{(n)}\right|\left\|f(a_1 + x_j^{(1)}, ..., a_n + x_j^{(n)}) - f(a_1, ..., a_n)\right\|\right)^q\right)^{1/q}\\ &\leq C\prod_{k=1}^{n}\sup_{\varphi\in B_{X_k^*}}\left(\sum_{j=1}^{m}\left(\left|b_j^{(k)}\right|\left|\varphi(x_j^{(k)})\right|\right)^{q_k}\right)^{1/q_k}\end{aligned}$$

for every positive integer m, $(x_j^{(k)}, b_j^{(k)}) \in X_k \times \mathbb{K}$, with $(j, k) \in \{1, ..., m\} \times \{1, ..., n\}$.

As pointed in [52], inequality (4.4) arises the curious idea of weighted summability: each $x_j^{(k)}$ is interpreted as having a "weight" $b_j^{(k)}$ and in this context the respective sum

$$\left\|f(a_1 + x_j^{(1)}, ..., a_n + x_j^{(n)}) - f(a_1, ..., a_n)\right\|$$

inherits a weight $\left|b_j^{(1)} \cdot \cdots \cdot b_j^{(n)}\right|$.

As it is shown in [15], the unified PDT (UPDT) immediately recovers several known Pietsch-type theorems. However, in at least one important situation (the PDT for dominated multilinear mappings), the respective PDT is not straightforwardly obtained from the UPDT from [15]. In fact, as pointed in [52], the structural difference between (4.2) and (4.3) is an obstacle to recover some domination theorems as Theorem 4.2. The same deficiency of the (general) UPDT will be clear in Section 4.3.

In the next section the approach of [52] is translated to a more abstract setting and the final result shows that Theorem 4.2 holds in a very general context. Some applications are given in order to show the reach of this generalization.

4.1. **The full general Pietsch Domination Theorem.** In this section we prove a quite general PDT which seems to delimit the possibilities of such kind of result. The procedure is an abstraction of the main result of [52]. It is curious the fact that the Unified Pietsch Domination Theorem from [15] does not use Pietsch's original argument, but this more general version, as in [52], uses precisely Pietsch's original approach in an abstract disguise.

The main tool of our argument, as in Pietsch's original proof of the linear case, is a Lemma by Ky Fan.

Lemma 4.3 (Ky Fan)**.** *Let K be a compact Hausdorff topological space and $\mathcal{F}$ be a concave family of functions $f : K \to \mathbb{R}$ which are convex and lower semicontinuous. If for each $f \in \mathcal{F}$ there is a $x_f \in K$ so that $f(x_f) \leq 0$, then there is a $x_0 \in K$ such that $f(x_0) \leq 0$ for every $f \in \mathcal{F}$.*

Let $X_1, ..., X_n$, Y and $E_1, ..., E_r$ be (arbitrary) non-void sets, $\mathcal{H}$ be a family of mappings from $X_1 \times \cdots \times X_n$ to Y . Let also $K_1, .., K_t$ be compact Hausdorff topological spaces, $G_1, ..., G_t$ be Banach spaces and suppose that the maps

$$\left\{ \begin{array}{l} R_j\colon K_j \times E_1 \times \cdots \times E_r \times G_j \longrightarrow [0,\infty),\ j = 1, ..., t \\ S\colon \mathcal{H} \times E_1 \times \cdots \times E_r \times G_1 \times \cdots \times G_t \longrightarrow [0,\infty) \end{array} \right.$$

satisfy:
(1) For each $x^{(l)} \in E_l$ and $b \in G_j$, with $(j, l) \in \{1, ..., t\} \times \{1, ..., r\}$ the mapping

$(R_j)_{x^{(1)},...,x^{(r)},b} : K_j \longrightarrow [0,\infty)$ defined by $(R_j)_{x^{(1)},...,x^{(r)},b}(\varphi) = R_j(\varphi, x^{(1)}, ..., x^{(r)}, b)$ is continuous.
(2) The following inequalities hold:

$$\left\{ \begin{array}{l} R_j(\varphi, x^{(1)}, ..., x^{(r)}, \eta_j b^{(j)}) \leq \eta_j R_j\left(\varphi, x^{(1)}, ..., x^{(r)}, b^{(j)}\right) \\ S(f, x^{(1)}, ..., x^{(r)}, \alpha_1 b^{(1)}, ..., \alpha_t b^{(t)}) \geq \alpha_1 ... \alpha_t S(f, x^{(1)}, ..., x^{(r)}, b^{(1)}, ..., b^{(t)}) \end{array} \right. \tag{4.5}$$

for every $\varphi \in K_j, x^{(l)} \in E_l$ (with $l = 1, ..., r$), $0 \leq \eta_j, \alpha_j \leq 1$, $b_j \in G_j$, with $j = 1, ..., t$ and $f \in \mathcal{H}$.

Definition 4.4. *If $0 < p_1, ..., p_t, p < \infty$, with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_t}$, a mapping $f : X_1 \times \cdots \times X_n \to Y$ in $\mathcal{H}$ is said to be $R_1, ..., R_t$-S-abstract $(p_1, ..., p_t)$-summing if there is a constant $C > 0$ so that*

$$\left(\sum_{j=1}^{m} S(f, x_j^{(1)}, ..., x_j^{(r)}, b_j^{(1)}, ..., b_j^{(t)})^p\right)^{\frac{1}{p}} \leq C \prod_{k=1}^{t} \sup_{\varphi \in K_k} \left(\sum_{j=1}^{m} R_k\left(\varphi, x_j^{(1)}, ..., x_j^{(r)}, b_j^{(k)}\right)^{p_k}\right)^{\frac{1}{p_k}} \tag{4.6}$$

for all $x_1^{(s)}, \ldots, x_m^{(s)} \in E_s$, $b_1^{(l)}, \ldots, b_m^{(l)} \in G_l$, $m \in \mathbb{N}$ and $(s, l) \in \{1, ..., r\} \times \{1, ..., t\}$.

The proof mimics the steps of the particular case proved in [52], and hence we omit some details. Due the more abstract environment, the new proof has extra technicalities but just in the final part of the proof a more important care will be needed when dealing with the parameter β.

As in the proof of [52], we need the following lemma (see [32, Page 17]):

Lemma 4.5. *Let $0 < p_1, ..., p_n, p < \infty$ be so that $1/p = \sum_{j=1}^{n} 1/p_j$. Then*

$$\frac{1}{p}\prod_{j=1}^{n} q_j^p \leq \sum_{j=1}^{n} \frac{1}{p_j} q_j^{p_j}$$

regardless of the choices of $q_1, .., q_n \geq 0$.

Now we are ready to prove the aforementioned theorem:

Theorem 4.6. *A map $f \in \mathcal{H}$ is $R_1, ..., R_t$-S-abstract $(p_1, ..., p_t)$-summing if and only if there is a constant $C > 0$ and Borel probability measures μ_j on K_j such that*

$$S(f, x^{(1)}, ..., x^{(r)}, b^{(1)}, ..., b^{(t)}) \leq C \prod_{j=1}^{t} \left(\int_{K_j} R_j \left(\varphi, x^{(1)}, ..., x^{(r)}, b^{(j)}\right)^{p_j} d\mu_j \right)^{1/p_j} \tag{4.7}$$

for all $x^{(l)} \in E_l$, $l = 1, ..., r$ and $b^{(j)} \in G_j$, with $j = 1, ..., t$.

Proof. One direction is canonical and we omit. Let us suppose that $f \in \mathcal{H}$ is $R_1, ..., R_t$-S-abstract $(p_1, ..., p_t)$-summing. Consider the compact sets $P(K_k)$ of the probability measures in $C(K_k)^*$, for all $k = 1, ..., t$. For each $(x_j^{(l)})_{j=1}^m$ in E_l and $(b_j^{(s)})_{j=1}^m$ in G_s, with $(s, l) \in \{1, ..., t\} \times \{1, ..., r\}$, let

$$g = g_{(x_j^{(l)})_{j=1}^m, (b_j^{(s)})_{j=1}^m, (s,l) \in \{1,...,t\} \times \{1,...,r\}} : P(K_1) \times \cdots \times P(K_t) \to \mathbb{R}$$

be defined by

$$g\left((\mu_j)_{j=1}^t\right) =$$
$$= \sum_{j=1}^m \left[\frac{1}{p} S(f, x_j^{(1)}, ..., x_j^{(r)}, b_j^{(1)}, ..., b_j^{(t)})^p - C^p \sum_{k=1}^t \frac{1}{p_k} \int_{K_k} R_k \left(\varphi, x_j^{(1)}, ..., x_j^{(r)}, b_j^{(k)}\right)^{p_k} d\mu_k \right].$$

As usual, the family $\mathcal{F}$ of all such g's is concave and one can also easily prove that every $g \in \mathcal{F}$ is convex and continuous. Besides, for each $g \in \mathcal{F}$ there are measures $\mu_j^g \in P(K_j)$, $j = 1, ..., t$, such that

$$g(\mu_1^g, ..., \mu_t^g) \leq 0.$$

In fact, using the compactness of each K_k $(k = 1, ..., t)$, the continuity of $(R_k)_{x_j^{(1)}, ..., x_j^{(r)}, b_j^{(k)}}$, there are $\varphi_k \in K_k$ so that

$$\sum_{j=1}^m R_k \left(\varphi_k, x_j^{(1)}, ..., x_j^{(r)}, b_j^{(k)}\right)^{p_k} = \sup_{\varphi \in K_k} \sum_{j=1}^m R_k \left(\varphi, x_j^{(1)}, ..., x_j^{(r)}, b_j^{(k)}\right)^{p_k}.$$

Now, with the Dirac measures $\mu_k^g = \delta_{\varphi_k}$, $k = 1, ..., t$, and Lemma 4.5 we get

$$g(\mu_1^g, ..., \mu_t^g) \leq 0.$$

So, Ky Fan's Lemma asserts that there are $\overline{\mu_j} \in P(K_j)$, $j = 1, ..., t$, so that

$$g(\overline{\mu_1}, ..., \overline{\mu_t}) \leq 0$$

for all $g \in \mathcal{F}$. Hence

$$\sum_{j=1}^m \left[\frac{1}{p} S(f, x_j^{(1)}, ..., x_j^{(r)}, b_j^{(1)}, ..., b_j^{(t)})^p \right] - C^p \sum_{k=1}^t \frac{1}{p_k} \int_{K_k} \sum_{j=1}^m R_k \left(\varphi, x_j^{(1)}, ..., x_j^{(r)} b_j^{(k)}\right)^{p_k} d\overline{\mu_k} \leq 0$$

and from the particular case $m = 1$ we obtain

$$\frac{1}{p} S(f, x^{(1)}, ..., x^{(r)}, b^{(1)}, ..., b^{(t)})^p \leq C^p \sum_{k=1}^t \frac{1}{p_k} \int_{K_k} R_k \left(\varphi, x^{(1)}, ..., x^{(r)}, b^{(k)}\right)^{p_k} d\overline{\mu_k}. \tag{4.8}$$

If $x^{(1)},...,x^{(r)},b^{(1)},...,b^{(t)}$ are given and, for $k=1,...,t$, define

$$\tau_k := \left(\int_{K_k} R_k\left(\varphi, x^{(1)},...,x^{(r)},b^{(k)}\right)^{p_k} d\overline{\mu_k}\right)^{1/p_k}.$$

If $\tau_k = 0$ for every k then, the result is immediate. Let us now suppose that τ_j is not zero for some $j \in \{1,...,t\}$. Consider

$$V = \{j \in \{1,..,t\}; \tau_j \neq 0\}$$

and $\beta > 0$ big enough to get

$$0 < \left(\tau_j \beta^{\frac{1}{pp_j}}\right)^{-1} < 1 \text{ for every } j \in V. \tag{4.9}$$

The above condition is necessary in view of (4.5). Consider, also,

$$\vartheta_j = \begin{cases} \left(\tau_j \beta^{\frac{1}{pp_j}}\right)^{-1} \text{ if } j \in V \\ 1 \text{ if } j \notin V. \end{cases}$$

Thus, since $0 < \vartheta_j \leq 1$, we have

$$\begin{aligned}
\frac{1}{p} S(f, x^{(1)},...,x^{(r)}, \vartheta_1 b^{(1)},...,\vartheta_t b^{(t)})^p &\leq C^p \sum_{k=1}^{t} \frac{1}{p_k} \int_{K_k} R_k\left(\varphi, x^{(1)},...,x^{(r)}, \vartheta_k b^{(k)}\right)^{p_k} d\overline{\mu_k} \\
&\leq C^p \sum_{k \in V} \frac{1}{p_k} \left(\tau_k \beta^{\frac{1}{pp_k}}\right)^{-p_k} \tau_k^{p_k} \\
&\leq \frac{C^p}{p} \frac{1}{\beta^{\frac{1}{p}}}
\end{aligned}$$

and

$$S(f, x^{(1)},...,x^{(r)}, b^{(1)},...,b^{(t)})^p \leq C^p \beta^{\left(\sum_{j \in V} 1/p_j\right) - 1/p} \prod_{j \in V} \tau_j^p. \tag{4.10}$$

If $V \neq \{1,...,t\}$, then

$$\frac{1}{p} - \sum_{j \in V} \frac{1}{p_j} > 0.$$

Note that it is possible to make $\beta \to \infty$ in (4.10), since it does not contradict (4.9); so we get

$$S(f, x^{(1)},...,x^{(r)}, b^{(1)},...,b^{(t)})^p = 0$$

and we again reach (4.7). The case $V = \{1,...,t\}$ is immediate. □

4.2. **Application: The (general) Unified PDT and the case of dominated multilinear mappings.** By choosing $r = t = n = 1$ in Theorem 4.6 we obtain an improvement of the Unified Pietsch Domination Theorem from [15]. In fact, we obtain precisely [53, Theorem 2.1] which is essentially the general unified PDT (we just need to repeat the trick used in [53, Theorem 3.1]).

It is interesting to note that, in the case $n > 1$, the trick used in [53, Theorem 3.1] is essentially what emerges the notion of weighted summability. In resume, this trick works perfectly for $n = 1$, but for other cases it forces us to deal with weighted summability. So, one shall not expect for the possible relaxation of conditions (4.5) for the validity of Theorem 4.6.

As pointed out in the introduction, contrary to what happens in [15], our theorem straightforwardly recovers the domination theorem for $(q_1, ..., q_n)$-dominated n-linear mappings (with $1/q = 1/q_1 + \cdots + 1/q_n$). In fact, we just need to choose

$$\begin{cases} t = n \\ G_j = X_j \text{ and } K_j = B_{X_j^*} \text{ for all } j = 1, ..., n \\ E_j = \mathbb{K}, j = 1, ..., r \\ \mathcal{H} = \mathcal{L}(X_1, ..., X_n; Y) \\ p_j = q_j \text{ for all } j = 1, ..., n \\ S(T, x^{(1)}, ..., x^{(r)}, b^{(1)}, ..., b^{(n)}) = \left\| T(b^{(1)}, ..., b^{(n)}) \right\| \\ R_k(\varphi, x^{(1)}, ..., x^{(r)}, b^{(k)}) = \left| \varphi(b^{(k)}) \right| \text{ for all } k = 1, ..., n. \end{cases}$$

So, with these choices, T is $R_1, .., R_n$-S abstract $(q_1, ..., q_n)$-summing precisely when T is $(q_1, ..., q_n)$-dominated. In this case Theorem 4.6 tells us that there is a constant $C > 0$ and there are measures μ_k on K_k, $k = 1, ..., n$, so that

$$S(T, x^{(1)}, ..., x^{(r)}, b^{(1)}, ..., b^{(n)}) \leq C \prod_{k=1}^{n} \left(\int_{K_k} R_k \left(\varphi, x^{(1)}, ..., x^{(r)}, b^{(k)} \right)^{q_k} d\mu_k \right)^{\frac{1}{q_k}},$$

i.e.,

$$\left\| T(b^{(1)}, ..., b^{(n)}) \right\| \leq C \prod_{k=1}^{n} \left(\int_{K_k} \left| \varphi(b^{(k)}) \right|^{q_k} d\mu_k \right)^{\frac{1}{q_k}}.$$

4.3. **Application: The PDT for Cohen strongly q-summing operators .** The class of Cohen strongly q-summing multilinear operators was introduced by D. Achour and L. Mezrag in [1]. Let $1 < q < \infty$ and $X_1, ..., X_n, Y$ arbitrary Banach spaces. If $q > 1$, then q^* denotes the real number satisfying $1/q + 1/q^* = 1$. A continuous n-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is Cohen strongly q-summing if and only if there is a constant $C > 0$ such that for any positive integer m, $x_1^{(j)}, ..., x_m^{(j)}$ in X_j $(j = 1, ..., n)$ and any $y_1^*, ..., y_m^*$ in Y^*, the following inequality hold:

$$\sum_{i=1}^{m} \left| y_i^* \left(T(x_i^{(1)}, ..., x_i^{(n)}) \right) \right| \leq C \left(\sum_{i=1}^{m} \prod_{j=1}^{n} \left\| x_i^{(j)} \right\|^q \right)^{1/q} \sup_{y^{**} \in B_{Y^{**}}} \left(\sum_{i=1}^{m} |y^{**}(y_i^*)|^{q^*} \right)^{1/q^*}.$$

In the same paper the authors also prove the following Pietsch-type theorem:

Theorem 4.7 (Achour-Mezrag). *A continuous n-linear mapping $T : X_1 \times \cdots \times X_n \to Y$ is Cohen strongly q-summing if and only if there is a constant $C > 0$ and a probability measure μ on $B_{Y^{**}}$ so that for all $(x^{(1)}, ..., x^{(n)}, y^*)$ in $X_1 \times \cdots \times X_n \times Y^*$ the inequality*

$$\left| y^* \left(T(x^{(1)}, ..., x^{(n)}) \right) \right| \leq C \left(\prod_{k=1}^{n} \left\| x^{(k)} \right\| \right) \left(\int_{B_{Y^{**}}} |y^{**}(y^*)|^{q^*} d\mu \right)^{\frac{1}{q^*}} \tag{4.11}$$

is valid.

Note that by choosing the parameters

$$\begin{cases} t = 2 \text{ and } r = n \\ E_i = X_i \text{ for all } i = 1, ..., n \\ K_1 = B_{X_1^* \times \cdots \times X_n^*} \text{ and } K_2 = B_{Y^{**}} \\ G_1 = \mathbb{K} \text{ and } G_2 = Y^* \\ \mathcal{H} = \mathcal{L}(X_1, ..., X_n; Y) \\ p = 1, \;\; p_1 = q \text{ and } p_2 = q^* \\ S(T, x^{(1)}, ..., x^{(n)}, b, y^*) = \left| y^* \left(T(x^{(1)}, ..., x^{(n)} \right) \right| \\ R_1(\varphi, x^{(1)}, ..., x^{(n)}, b) = \left\| x^{(1)} \right\| \cdots \left\| x^{(n)} \right\| \\ R_2(\varphi, x^{(1)}, ..., x^{(n)}, y^*) = |\varphi(y^*)| \end{cases}$$

we can easily conclude that $T : X_1 \times \cdots \times X_n \to Y$ is Cohen strongly q-summing if and only if T is R_1, R_2-S abstract (q, q^*)-summing. Theorem 4.6 tells us that T is R_1, R_2-S abstract (q, q^*)-summing if and only if there is a $C > 0$ and there are probability measures μ_k in K_k, $k = 1, 2$, such that

$$S(T, x^{(1)}, ..., x^{(n)}, b, y^*) \leq C \left(\int_{K_1} R_1 \left(\varphi, x^{(1)}, ..., x^{(n)}, b \right)^q d\mu_1 \right)^{\frac{1}{q}} \left(\int_{K_2} R_2 \left(\varphi, x^{(1)}, ..., x^{(n)}, y^* \right)^{q^*} d\mu_2 \right)^{\frac{1}{q^*}},$$

i.e.,

$$\begin{aligned} \left| y^* \left(T(x^{(1)}, ..., x^{(n)}) \right) \right| &\leq C \left(\int_{B_{X_1^* \times \cdots \times X_n^*}} \left(\left\| x^{(1)} \right\| \cdots \left\| x^{(n)} \right\| \right)^q \; d\mu_1 \right)^{\frac{1}{q}} \left(\int_{B_{Y^{**}}} |\varphi(y^*)|^{q^*} d\mu_2 \right)^{\frac{1}{q^*}} \\ &= C \left\| x^{(1)} \right\| ... \left\| x^{(n)} \right\| \left(\int_{B_{Y^{**}}} |\varphi(y^*)|^{q^*} d\mu_2 \right)^{\frac{1}{q^*}} \end{aligned}$$

and we recover (4.11) regardless of the choice of the positive integer m and $x^{(k)} \in X_k$, $k = 1, ..., n$.

5. Weighted summability

The notion of weighted summability (see the comments just after Theorem 4.2) emerged from the paper [52] as a natural concept when we were dealing with problem (4.3).

In this section we observe that this concept in fact emerges in more abstract situations and seems to be unavoidable in further developments of the nonlinear theory.

Let $0 < q_1, ..., q_n < \infty$, $1/q = \sum_{j=1}^{n} 1/q_j$, $X_1, ..., X_n$ be Banach spaces and

$$A : Map(X_1, ..., X_n; Y) \times X_1 \times \cdots \times X_n \to [0, \infty)$$

be an arbitrary map. Let us say that $f \in Map(X_1, ..., X_n; Y)$ is A-$(q_1, ..., q_n)$-dominated if there is a constant $C > 0$ so that

(5.1)
$$A(f,x^{(1)},...,x^{(n)}) \leq C\left(\int_{B_{X_1^*}} \left|\varphi(x^{(1)})\right|^{q_1} d\mu_1\right)^{\frac{1}{q_1}} \cdot \cdots \cdot \left(\int_{B_{X_n^*}} \left|\varphi(x^{(n)})\right|^{q_n} d\mu_k\right)^{\frac{1}{q_n}},$$

regardless of the choice of the positive integer m and $x^{(k)} \in X_k$, $k = 1,...,n$.

In fact, more abstract maps could be used in the right-hand side of (5.1). However, since our intention is illustrative rather than exhaustive, we prefer to deal with this more simple case.

Theorem 5.1. *An arbitrary map $f \in Map(X_1,...,X_n;Y)$ is A-$(q_1,...,q_n)$-dominated if there exists $C>0$ such that*
(5.2)
$$\left(\sum_{j=1}^{m}\left(\left|b_j^{(1)}...b_j^{(n)}\right| A(f,x_j^{(1)},...,x_j^{(n)})\right)^q\right)^{\frac{1}{q}} \leq C\prod_{k=1}^{n}\sup_{\varphi\in B_{X_k^*}}\left(\sum_{j=1}^{m}\left(\left|b_j^{(k)}\right|\left|\varphi(x_j^{(k)})\right|\right)^{q_k}\right)^{1/q_k}$$

for every positive integer m, $(x_j^{(k)}, b_j^{(k)}) \in X_k \times \mathbb{K}$, with $(j,k) \in \{1,...,m\}\times\{1,...,n\}$.

Proof. Choosing the parameters
$$\begin{cases} r=t=n \\ E_j = X_j \text{ and } G_j = \mathbb{K} \text{ for all } j=1,...,n \\ K_j = B_{X_j^*} \text{ for all } j=1,...,n \\ \mathcal{H} = Map(X_1,...,X_n;Y) \\ p=q \text{ and } p_j = q_j \text{ for all } j=1,...,n \\ S(f,x^{(1)},...,x^{(n)},b^{(1)},...,b^{(n)}) = \left|b^{(1)}...b^{(n)}\right| A(f,x^{(1)},...,x^{(n)}) \\ R_k(\varphi,x^{(1)},...,x^{(n)},b^{(k)}) = \left|b^{(k)}\right|\left|\varphi(x^{(k)})\right| \text{ for all } k=1,...,n. \end{cases}$$

we easily conclude that (5.2) holds if and only if f is $R_1,..,R_n$-S abstract $(q_1,...,q_n)$-summing. In this case Theorem 4.6 tells us that there is a constant $C>0$ and there are measures μ_k on K_k, $k=1,...,n$, such that

$$S(T,x^{(1)},...,x^{(n)},b^{(1)},...,b^{(n)}) \leq C\prod_{k=1}^{n}\left(\int_{K_k} R_k\left(\varphi,x^{(1)},...,x^{(n)},b^{(k)}\right)^{q_k} d\mu_k\right)^{\frac{1}{q_k}},$$

i.e.,

$$\left|b^{(1)}...b^{(n)}\right| A(f,x^{(1)},...,x^{(n)}) \leq C\prod_{k=1}^{n}\left(\int_{B_{X_k^*}}\left(\left|b^{(k)}\right|\left|\varphi(x^{(k)})\right|\right)^{q_k} d\mu_k\right)^{\frac{1}{q_k}},$$

for all $(x^{(k)}, b^{(k)}) \in X_k \times \mathbb{K}$, $k=1,...,n$, and we readily obtain (5.1). $\square$

Remark 5.2. *As we have mentioned before, the procedure of this last section is illustrative. The interested reader can easily find a characterization similar to Theorem 5.1 in the full abstract context of Definition 4.4.*

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